Implications and data mining

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Abstract

In this work we review some properties of various implication types. We watch both theoretical and practical aspects of implication usage. We study several generalizations of the classical implication which are used in fuzzy logic. We study notable kinds of modus ponens which are exploited in data mining technics. While we do not aim to provide new theorems, still we propose new usage of multitudination in fuzzy reasoning.

Introduction

Implications express one-way relations and thus they are a more general tool than symmetrical functions like equivalences and correlations. It is widely used in both theory and applications. The implication in classical propositional logic can be defined by the table below:

a	b	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

Generally, its work is determined by axioms and modus ponens. If we have valid a and $a \Rightarrow b$ then we can deduce b:

$$\frac{a,\, a \Rightarrow b}{b}$$

Fuzzy logic and various practical applications use several generalizations of the ideas presented above. Our investigations are focused in models based on [0, 1] interval. It is the main case for applications.

Implications in pure mathematics

Fuzzy logics make usage of implications of several kinds [13, 16, 20]. They are usually called S-implications, R-implications, and A-implications. Let us recall standard axioms for fuzzy implicators, i.e. fuzzy implication functions [2, 8].

I is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$.

Ax. 1 If $(x \leq y)$ then $(I(x,r) \geq I(y,r))$
Ax. 2 If $(r \leq s)$ then $I(x,r) \leq I(x,s)$
Ax. 3 $I(0,r) = 1$
Ax. 4 $I(1,r) = r$
Ax. 5 $I(x, I(y,r)) = I(y, I(x,r))$
Ax. 6 If $(x \leq r)$ then $I(x,r) = 1$
Ax. 7 If $(x,r) = 1$ then $(x \leq r)$
Ax. 8 $I(x,r) \geq r$
Ax. 9 $I(x,x) = 1$
Ax. 10 I is continuous function

Note that not all of the listed axioms are held by all functions which are called implicators.

The first class of implications which is mentioned above, is of S-implications. They are gained by combinations of conjunctions (t-norms), disjunctions (t-conorms), and negations (strong negators). They are, for example:

$$\begin{aligned} S_1 &= (\neg a) \vee b \\ S_2 &= (\neg a) \vee (a \wedge b) \end{aligned}$$

where $\neg$ is negation, $\vee$ is disjunction, $\wedge$ is conjunction. The first example (S_1) is S-implication in a narrow sense. The second example (S_2) is so called QL-implication.

The second class of implications are R-implications. They are gained by additional usage of quantifiers when we want to express them in the language of first order arithmetics. It is, for example, so called residuation of conjunctions (t-norms):

$$R_*(a,b) = max\{i \in [0,\, 1],\, (a * i) \leq b\}$$

where $*$ is respective t-norm. It is R-implication in a narrow sense. Herein, the max operator makes usage of quantifiers in its expression in a standard way: "for every x, if the respective condition is met then $x \leq i$." Each of S implications can be, of course, defined as an R-implication. It should be noted that all the implications are relations and hence they are used as connectives - they are not quantifiers themselves.

The third class of implications are A-implications [7]. This class is the general one. We just set specific axioms which the considered implication must hold. Each implication can be defined in this way. One implicator which is not a member of R-implications, is Yager implication:

$$A_Y(a, b) = b^a$$

There are several generalizations of implication connectives. Among others, they are co-implications and residual operators of generalized norms.

Coimplicators [3, 6] are an analogy to implicators in a way they are used for sentence proving. While implicators are used for proving validity, coimplicators are used for proving nonvalidity. Classical coimplicator can be defined by the table below:

a	b	$a \overset{\circ}{\Rightarrow} b$
0	0	0
0	1	1
1	0	0
1	1	0

Residual coimplicators (CR) are defined as

$$CR_\circ(a, b) = min\{j \in [0, 1], (a \circ j) \geq b\}$$

where $\circ$ is respective t-conorm.

Residual operators can be defined for generalized norm-like functions. Ones of the most studied cases are uninorm utilizations [1, 10, 17]. Uninorms generalize t-norms and t-conorms by setting their neutral element into arbitrarily chosen place in [0, 1] interval. We can define both residual implicators and coimplicators for given uninorms [4]. The definitions are formally the same as are given above for t-norms and co-norms. The only difference is that we use an uninorm in place of a t-norm (of a t-conorm respectively).

Implications in applied mathematics

Practical applications usually require usage of continuous functions. Therefore, residual implications have constrained usage. Nevertheless, there are situations where we can repair implication behavior into overall continuity.

There are two main ways of fuzzy logic usage on data processing. First, it is data mining. We extract rules from available data during data mining technics processing. Second, it is rule-based decision making. We use available rules during control technics work.

Since we are interested in data mining, we do not exploit rule-based systems at this work. We can just note that one seriously used idea is of combining functions [11]. It is based on uninorm usage, where experts stress either positive (higher than neutral element) notions or negative (lower than neutral element) notions. The notions are combined by uninorm actions. Another widely used method is generalized modus ponens [5, 9, 21].

Data mining is a class of methods for gaining features which are present in data. The features can be wanted for various reasons. One of them is the use of the features in subsequent decision making.

There are plenty of methods how to exploit data. Some ways are effectual in learning on data, but they have difficulties to interpret the learnt rules, e.g. neural networks. Some ways are effectual in interpretations of learnt rules, but they have difficulties in learning, e.g. structures of conditional independence.

There are two other ways: statistical and logical. Methods of statistics make use of abducting reasoning. It is substitution for modus ponens. Methods of logic make use of generalized quantifiers. They are substitution for implication connectives.

Classical quantifiers require absolute, coarse-grained conditions, i.e. some given property holds on every case or there is at least one case where some given property holds. Generalized quantifiers lie between the limits of classical quantifiers. They can be "there are many", "we have at least p percent of" and other ones. Such quantifiers deals with single formulae.

Generalized quantifiers are frequently defined on formula pairs applied on data collections as well. Let us have, for example, fourfold data table for two formulae on some (e.g. experimental or business) data. Our data cases can be e.g. shopping baskets with formulae made on their contents (products bought by individual consumers). The fourfold table for formulae ϕ and ψ is

$\phi \setminus \psi$	1	0
1	a	b
0	c	d

where a stands for number of cases where both ϕ and ψ formulae hold on the used data collection individuals. The letters b, c and d are defined analogically, e.g. c stands for cases where ϕ does not hold but ψ does hold. Here, we can have other interesting quantifiers: "ψ holds for many cases where ϕ holds" and other ones. This is used for so called multitudinal quantifiers that are described below in this text.

The abduction looks for explanation formulae [18, 19]. It takes place when we have a data valid formula, but the formula is not deductible. In such a case, we try to find the most reasonable formula to explain the data valid formula. One approach how to do it, is Bayesian method. The data valid formula acts as a posterior knowledge. The theory we can deal with, is class of a prior knowledge. Then, search for the most reasonable explanatory formula is according to the Bayes formula:

$$P(A_i|\, B) = \frac{P(B|\, A_i) * P(A_i)}{\sum_j [P(B|\, A_j) * P(A_j)]}$$

where $*$ is multiplication. We look for A_i with the greatest $P(A_i|\, B)$. The resulted A_i is both an explanation and an instruction for direction of further data handling.

The quantifier testing is a method when we systematically construct formulae and evaluate chosen quantifiers on them [12, 14, 15]. We start from the most power formulae with respect to the chosen quantifiers. The quantifiers mimic connective functions from reasoning point of view. We can choose what is the best kind of connectives for our investigations. In case of implication, multitudinal quantifiers are a reasonable choice. Definition of the multitudinality of a quantifier Q on crisp data is given below:

$\phi \setminus \psi$	1	0
1	a	b
0	c	d

The quantifier must fulfill two conditions:
I) if $a_1 \geq a_2$ then $Q(\phi_1, \psi) \geq Q(\phi_2, \psi)$;
II) if $b_1 \geq b_2$ then $Q(\phi, \psi_2) \geq Q(\phi, \psi_1)$.
They can be e.g. $a/(a + b)$.

The multitudinal quantifiers are a compromise between pure implications and pure conjunctions. Implications $(a + c + d)/(a + b + c + d)$ are used for upper approximations of functions. Conjunctions $(a)/(a + b + c + d)$ are used for lower approximation of functions.

Multitudinality can be viewed as an use of "realistic implication" or "promising causality", contrary to standard implication usage. Nevertheless, it is just a hope for a causality, though a greater hope than the case of implication. The multitudinality of a quantifier can be used for modus ponens. It is explained below.

Again, we should note that there is a substantial difference between quantifiers and connectives. First, connectives fill in data tables. They produce truth values for particular individuals, e.g. particular shopping baskets. Second, quantifiers produce single overall numbers (truth values) on whole data tables, e.g. on consumers' habits.

We propose one possible generalization of validity of formulae and of modus ponens in many valued logic. Having a pair of formulae with their validities $(\phi,\, val(\phi))$, $(\phi \Rightarrow \psi,\, val(\phi \Rightarrow \psi))$, we can deduce $(\psi,\, t(val(\phi), val(\phi \Rightarrow \psi)))$, where t() is respective t-norm. A multitudinal quantifier can be used in place of implication formula. However usage of $Q(\phi, \psi)$ is to be limited for formulae of structure "validity of ψ is greater than or equal to c" where $c \leq val(\phi)$. In such a way, we can keep ourselves away from unreasonably low or high approximations.

One rather used type of implication is the one based on idempotent t-norm. The idempotent t-norm is on two values a and b is just $min(a,\, b)$. Its residuated implication is ether 1 for cases where $a \leq b$, or b for the other cases. Logic based on these connectives is called Goedel logic.

First, it is fast for computing and therefore it is frequently used in practical applications. Second, Goedel logic and intuitionistic logic share some of their models - Heyting algebras, i.e. bounded lattices with residuated implications. Heyting algebras are used in category theory as classifier objects of topoi categories and they are used in some interpretations of quantum physics theory [22]. Hence, idempotency is employed in varius areas from fast computations to theoretical physics.

Conclusion

We have reviewed several aspects of the implication connective and its generalizations. Our intention was directed both to theoretical aspects and to practical applications. Together with it, we have made a proposal of a new method for fuzzy reasoning which is based on multitudinality. We hope that our work can serve as a modest intriguing point.

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